

GENERIC ANALYTIC POLYHEDRON WITH NON-COMPACT AUTOMORPHISM GROUP

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ABSTRACT. In this paper we prove the following rigidity theorem: a generic analytic polyhedron with non-compact automorphism group is biholomorphic to the product of a complex manifold with compact automorphism group and a polydisk. Moreover, this complex manifold and the dimension of this polydisk can be explicitly described in terms of the limit set of the automorphism group.

1. INTRODUCTION

Given a bounded domain $\Omega \subset \mathbb{C}^d$ let $\text{Aut}(\Omega)$ be the *automorphism group* of Ω , that is the group of biholomorphisms $f : \Omega \rightarrow \Omega$. The group $\text{Aut}(\Omega)$ has a Lie group structure compatible with the compact-open topology and $\text{Aut}(\Omega)$ acts properly on Ω . It is a long standing problem to characterize the domains Ω where $\partial\Omega$ has nice properties and $\text{Aut}(\Omega)$ is non-compact (see the survey [IK99]). One well known result along these lines is the Wong-Rosay ball theorem:

Theorem 1.1 (The Wong-Rosay Ball Theorem [Ros79, Won77]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain. Then $\text{Aut}(\Omega)$ is non-compact if and only if Ω is biholomorphic to the unit ball.*

In this paper we will consider a particular class of domains which are in some sense as far as possible from being strongly pseudoconvex:

Definition 1.2. A domain $\Omega \subset \mathbb{C}^d$ is called an *analytic polyhedron* if there exists a neighborhood U of Ω and holomorphic functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$ so that

$$\Omega = \{z \in U : |f_i(z)| < 1 \text{ for } i = 1, \dots, N\}.$$

An analytic polyhedron is called *generic* if we can choose the functions f_1, \dots, f_N so that whenever $\xi \in \partial\Omega$ and $|f_{i_1}(\xi)| = \dots = |f_{i_r}(\xi)| = 1$, then the vectors

$$\nabla f_{i_1}(\xi), \dots, \nabla f_{i_r}(\xi)$$

are \mathbb{C} -linearly independent. In this case we call f_1, \dots, f_N a *set of generic defining functions* for Ω .

The simplest example of an analytic polyhedron with non-compact automorphism group is the polydisk:

$$\Delta^r = \{(z_1, \dots, z_r) \in \mathbb{C}^r : |z_1| < 1, \dots, |z_r| < 1\}.$$

Additional examples can be constructed by taking a product of a polydisk and an analytic polyhedron. When Ω is convex, Kim showed that (up to biholomorphism)

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these are the only examples of generic analytic polyhedron with non-compact automorphism group:

Theorem 1.3. [Kim92] *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex generic analytic polyhedron and $\text{Aut}(\Omega)$ is non-compact. Then there exists some $r > 0$ and a convex domain $W \subset \mathbb{C}^{d-r}$ so that Ω is biholomorphic to $\Delta^r \times W$.*

In dimension two Kim, Krantz, and Spiro removed the convexity hypothesis and gave an explicit description of r and W :

Theorem 1.4. [KKS05] *Suppose $\Omega \subset \mathbb{C}^2$ is a bounded generic analytic polyhedron and $\text{Aut}(\Omega)$ is non-compact. Then:*

- (1) *If an automorphism orbit accumulates at a singular boundary point, then Ω is biholomorphic to Δ^2 ,*
- (2) *If an automorphism orbit accumulates at a smooth boundary point, then Ω is biholomorphic to the product of Δ and the maximal analytic variety in $\partial\Omega$ passing through the orbit accumulation point.*

Remark 1.5. Kim, Krantz, and Spiro's theorem generalized an earlier result of Kim and Pagino [KP01] and is related to a result of Fu and Wong [FW00]: any simply-connected domain in \mathbb{C}^2 with generic piecewise smooth Levi-flat boundary and non-compact automorphism group is biholomorphic to a bidisc.

In this paper we complete the characterization of bounded generic analytic polyhedron with non-compact automorphism group:

Theorem 1.6. *Suppose Ω is a bounded generic analytic polyhedron and $\text{Aut}(\Omega)$ is non-compact. Then there exists some $r > 0$ and a complex manifold W with $\text{Aut}(W)$ compact so that Ω is biholomorphic to $\Delta^r \times W$.*

As in the Kim, Krantz, and Spiro result we can explicitly describe r and W in terms of the limit set of $\text{Aut}(\Omega)$.

Definition 1.7. Suppose Ω is a bounded domain in \mathbb{C}^d . The *limit set* of $\text{Aut}(\Omega)$ is the set $\mathcal{L}(\Omega) \subset \partial\Omega$ of points $\xi \in \partial\Omega$ where there exists a sequence $\varphi_n \in \text{Aut}(\Omega)$ and a point $x \in \Omega$ with $\varphi_n(x) \rightarrow \xi$.

Remark 1.8. When Ω is a bounded domain $\text{Aut}(\Omega)$ acts properly on Ω and so the set $\mathcal{L}(\Omega)$ is non-empty if and only if $\text{Aut}(\Omega)$ is non-compact.

Suppose Ω is a generic analytic polyhedron and f_1, \dots, f_N is a set of generic defining functions for Ω . Then for a point $\xi \in \partial\Omega$ define

$$\mathcal{I}(\xi) = \{i : |f_i(\xi)| = 1\}$$

and

$$r(\xi) := \#\mathcal{I}(\xi).$$

Also let $\mathcal{F}(\xi) \subset \partial\Omega$ be the connected component of

$$\{\eta \in \partial\Omega : f_i(\eta) = f_i(\xi) \text{ if } i \in \mathcal{I}(\xi) \text{ and } |f_i(\eta)| < 1 \text{ if } i \notin \mathcal{I}(\xi)\}$$

which contains ξ . Then $\mathcal{F}(\xi)$ is the maximal analytic variety in $\partial\Omega$ passing through ξ . With this notation we will prove the following:

Theorem 1.9. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron. If $\xi \in \mathcal{L}(\Omega)$, then Ω is biholomorphic to $\Delta^{r(\xi)} \times \mathcal{F}(\xi)$.*

We can also explicitly describe the complex manifold W in the statement of Theorem 1.6.

Proposition 1.10. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron. If $\xi \in \mathcal{L}(\Omega)$ and*

$$r(\xi) = \max\{r(\eta) : \eta \in \mathcal{L}(\Omega)\},$$

then $\text{Aut}(\mathcal{F}(\xi))$ is compact.

Notice that Theorem 1.6 is an immediate consequence of Theorem 1.9 and Proposition 1.10.

1.1. Outline of the proof of Theorem 1.9 and Proposition 1.10: Like many of the results characterizing domains with large automorphism groups, a key step in our argument is rescaling. Historically rescaling has been successfully implemented only when the domain in question is

- (1) convex (see for instance [Fra89, Kim92, BP94, Gau97]),
- (2) in \mathbb{C}^2 (see for instance [BP88, BP98, KKS05, Ver09]), or
- (3) strongly pseudoconvex (see for instance [Pin91]).

In particular, none of the classical rescaling methods apply directly to analytic polyhedron. In the proof below, we make a rescaling argument work by constructing a holomorphic embedding $F : \Omega \hookrightarrow \Delta^M$, proving it has good properties, and then applying the rescaling method from the convex case to $F(\Omega) \subset \Delta^M$.

This embedding is constructed in an obvious way: if necessary we introduce additional “dummy” holomorphic functions $f_{N+1}, \dots, f_M : U \rightarrow \mathbb{C}$ so that:

- (1) for any $N+1 \leq i \leq M$, $f_i(\overline{\Omega}) \subset \Delta$,
- (2) for any $z, w \in \overline{\Omega}$ distinct there exists some $1 \leq i \leq M$ so that $f_i(z) \neq f_i(w)$,
- (3) for any point $z \in \overline{\Omega}$ we have

$$\text{Span}_{\mathbb{C}} \{\nabla f_1(z), \dots, \nabla f_M(z)\} = \mathbb{C}^d.$$

Then the map $F = (f_1, \dots, f_M) : U \rightarrow \mathbb{C}^M$ induces a holomorphic embedding $F : \Omega \hookrightarrow \Delta^M$.

Proving that this embedding has good properties is more involved. This is accomplished in Sections 4 and 5 where the main aim is to prove estimates for the Kobayashi metric and distance showing that F behaves like a quasi-isometric embedding.

Then given an unbounded sequence $\varphi_n \in \text{Aut}(\Omega)$ we consider the points $\overline{w}_n = F(\varphi_n(w_0)) \in \Delta^M$ and apply the rescaling method to the sequence $(\Delta^M, \overline{w}_n)$. This produces affine maps $\overline{A}_n \in \text{Aff}(\mathbb{C}^M)$ so that (up to reordering coordinates and passing to a subsequence) the sequence $\overline{A}_n(\Delta^M)$ converges in the local Hausdorff topology to $\mathcal{H}^r \times \Delta^{M-r}$ and $\overline{A}_n(\overline{w}_n)$ converges to a point $\overline{w}_\infty \in \mathcal{H}^r \times \Delta^{M-r}$.

Then using the estimates on the Kobayashi metric and distance, we will show that $(\overline{A}_n F \varphi_n) : \Omega \rightarrow \mathbb{C}^M$ is a normal family and after passing to a subsequence converges to a holomorphic embedding $\Phi : \Omega \rightarrow \mathcal{H}^r \times \Delta^{M-r}$. The final step in the proof of Theorem 1.9 is to show that the image of Φ has the form $\mathcal{H}^r \times W$ where W is biholomorphic to $\mathcal{F}(\xi)$.

To prove Proposition 1.10 we will use the classical theory of characteristic decompositions of analytic polyhedron to establish the following refinement of Theorem 1.9:

Theorem 1.11. (see Theorem 8.1 below) Suppose $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron. If $\xi \in \mathcal{L}(\Omega)$, then there exists a biholomorphism $\Phi : \Omega \rightarrow \Delta^{r(\xi)} \times \mathcal{F}(\xi)$ so that: if $\theta \in \text{Aut}(\mathcal{F}(\xi))$ and $\hat{\theta} = \Phi^{-1}(\text{id}, \theta)\Phi$ then

$$f_i(\hat{\theta}(z)) = f_i(z)$$

for all $i \in \mathcal{I}(\xi)$ and $z \in \Omega$.

Using this refinement it is straightforward to prove the contrapositive of Proposition 1.10: if $\xi \in \mathcal{L}(\Omega)$ and $\text{Aut}(\mathcal{F}(\xi))$ is non-compact, then there exists some $\eta \in \mathcal{L}(\Omega)$ with $r(\eta) > r(\xi)$.

1.2. Basic notation. We now fix some very basic notations.

- Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.
- Let $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.
- Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product on \mathbb{C}^d and for $z \in \mathbb{C}^d$ let $\|z\| := \langle z, z \rangle^{1/2}$.
- Given two open sets $\Omega_1 \subset \mathbb{C}^{d_1}$ and $\Omega_2 \subset \mathbb{C}^{d_2}$ let $\text{Hol}(\Omega_1, \Omega_2)$ be the space of holomorphic maps from Ω_1 to Ω_2 .
- Given two open sets $\mathcal{O}_1 \subset \mathbb{R}^{d_1}$, $\mathcal{O}_2 \subset \mathbb{R}^{d_2}$, a C^1 map $F : \mathcal{O}_1 \rightarrow \mathcal{O}_2$, and a point $x \in \mathcal{O}_1$ define the derivative $d(F)_x : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$ of F at x by

$$d(F)_x(v) := \left. \frac{d}{dt} \right|_{t=0} F(x + tv).$$

- Given a domain $\Omega \subset \mathbb{C}^d$ and a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ define $\nabla f(z) \in \mathbb{C}^d$ by

$$d(f)_z(v) = \langle v, \nabla f(z) \rangle \text{ for all } v \in \mathbb{C}^d.$$

2. THE KOBAYASHI METRIC AND RESCALING

In this section we recall some basic properties of the Kobayashi metric and distance. A more thorough discussion can be found in [Kob05] or [Aba89]. We will then discuss the rescaling method in the particular case of polydisks.

2.1. The Kobayashi metric and distance. Given a domain $\Omega \subset \mathbb{C}^d$ the (*infinitesimal*) *Kobayashi metric* is the pseudo-Finsler metric

$$k_\Omega(x; v) = \inf \{ |\zeta| : f \in \text{Hol}(\Delta, \Omega), f(0) = x, d(f)_0(\zeta) = v \}.$$

By a result of Royden [Roy71, Proposition 3] the Kobayashi metric is an upper semicontinuous function on $\Omega \times \mathbb{C}^d$. In particular if $\sigma : [a, b] \rightarrow \Omega$ is an absolutely continuous curve (as a map $[a, b] \rightarrow \mathbb{C}^d$), then the function

$$t \in [a, b] \rightarrow k_\Omega(\sigma(t); \sigma'(t))$$

is integrable and we can define the *length* of σ to be

$$\ell_\Omega(\sigma) = \int_a^b k_\Omega(\sigma(t); \sigma'(t)) dt.$$

One can then define the *Kobayashi pseudo-distance* to be

$$K_\Omega(x, y) = \inf \{ \ell_\Omega(\sigma) : \sigma : [a, b] \rightarrow \Omega \text{ is absolutely continuous,} \\ \text{with } \sigma(a) = x, \text{ and } \sigma(b) = y \}.$$

This definition is equivalent to the standard definition of K_Ω via analytic chains, see [Ven89, Theorem 3.1].

One of the most important properties of these constructions is the following distance decreasing property (which is immediate from the definitions):

Proposition 2.1. *Suppose $\Omega_1 \subset \mathbb{C}^{d_1}$ and $\Omega_2 \subset \mathbb{C}^{d_2}$ are domains and $f : \Omega_1 \rightarrow \Omega_2$ is a holomorphic map. Then*

$$k_{\Omega_2}(f(x); d(f)_x(v)) \leq k_{\Omega_1}(x; v)$$

and

$$K_{\Omega_2}(f(x), f(y)) \leq K_{\Omega_1}(x, y)$$

for all $x, y \in \Omega_1$ and $v \in \mathbb{C}^{d_1}$. In particular, when $\Omega \subset \mathbb{C}^d$ is a domain the group $\text{Aut}(\Omega)$ acts by isometries on the pseudo-metric space (Ω, K_Ω) .

2.2. The unit disk. Using the Schwarz lemma it is straightforward to see that the Kobayashi metric coincides with the Poincaré metric on the unit disk (at least up to a constant). In particular,

$$k_\Delta(x; v) = \frac{|v|}{1 - |x|^2}$$

and

$$K_\Delta(x, y) = \tanh^{-1} \left| \frac{x - y}{1 - x\bar{y}} \right|$$

for all $x, y \in \Delta$ and $v \in \mathbb{C}$.

This explicit description implies the following localization result:

Observation 2.2. Suppose $\xi \in \partial\Delta$ and U is a neighborhood of ξ . Then for any $\delta > 0$ there exists a neighborhood $V \Subset U$ so that

$$k_{U \cap \Delta}(x; v) \leq e^\delta k_\Delta(x; v)$$

and

$$K_{U \cap \Delta}(x, y) \leq e^\delta K_\Delta(x, y)$$

for all $x, y \in V \cap \Delta$ and $v \in \mathbb{C}$.

2.3. The Kobayashi metric on products. If

$$\Omega = \Omega_1 \times \cdots \times \Omega_k \subset \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k}$$

is a product of domains, it is straightforward to verify that

$$k_\Omega(x; v) \leq \max_{i=1, \dots, k} k_{\Omega_i}(x_i, v_i)$$

for all $x = (x_1, \dots, x_k) \in \Omega$ and $v = (v_1, \dots, v_k) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k}$. Using Proposition 2.1 and the natural projections, one also has

$$k_\Omega(x; v) \geq \max_{i=1, \dots, k} k_{\Omega_i}(x_i, v_i)$$

for all $x = (x_1, \dots, x_k) \in \Omega$ and $v = (v_1, \dots, v_k) \in \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_k}$. Thus we have the following:

Observation 2.3. With the notation above,

$$k_{\Omega}(x; v) = \max_{i=1, \dots, k} k_{\Omega_i}(x_i, v_i)$$

for all $x = (x_1, \dots, x_k) \in \Omega$ and $v = (v_1, \dots, v_k) \in \mathbb{C}^{d_1 + \dots + d_k}$.

2.4. The Kobayashi metric on analytic polyhedron. For analytic polyhedron the Kobayashi pseudo-distance is Cauchy complete:

Proposition 2.4. *Suppose that Ω is a bounded analytic polyhedron. Then (Ω, K_{Ω}) is a Cauchy complete metric space.*

Proof. Since Ω is bounded, (Ω, K_{Ω}) is a metric space, see for instance Corollaries 2.3.2 and 2.3.6 in [Aba89]. Now if $f : \Omega \rightarrow \Delta$ is holomorphic Proposition 2.1 implies that

$$K_{\Delta}(f(z), f(w)) \leq K_{\Omega}(z, w)$$

for all $z, w \in \Omega$. So, since Ω is an analytic polyhedron, the metric space (Ω, K_{Ω}) is proper (that is, bounded sets in (Ω, K_{Ω}) are relatively compact). But by definition (Ω, K_{Ω}) is a length space and so by the Hopf-Rinow theorem for length spaces, see for instance Corollary 3.8 in Chapter I of [BH99], (Ω, K_{Ω}) is Cauchy complete. \square

2.5. Rescaling polydisks. As mentioned in the introduction, we will use the rescaling method from the convex case in the proof of Theorem 1.9. In the case of polydisks, this procedure is very explicit and in this subsection we will describe only the observations we need. For a general discussion of rescaling methods see [Fra89, Pin91, Fra91, Kim04, KK08].

Suppose that $w_n = (w_n^{(1)}, \dots, w_n^{(r)})$ is a sequence in Δ^r and

$$\lim_{n \rightarrow \infty} w_n = \xi \in (\partial\Delta)^r.$$

Next consider affine map $A_n \in \text{Aff}(\mathbb{C}^r)$ given by

$$A_n(z_1, \dots, z_r) = \left(\lambda_n^{(1)} \left(z_1 - \frac{w_n^{(1)}}{|w_n^{(1)}|} \right), \dots, \lambda_n^{(r)} \left(z_r - \frac{w_n^{(r)}}{|w_n^{(r)}|} \right) \right)$$

where

$$\lambda_n^{(i)} = \frac{i |w_n^{(i)}|}{w_n^{(i)} (1 - |w_n^{(i)}|)}.$$

Then $A_n(w_n) = (i, \dots, i)$. Since

$$\lim_{n \rightarrow \infty} |w_n^{(i)}| = 1 \text{ for } 1 \leq i \leq r,$$

it is easy to see that $A_n(\Delta^r)$ converges in the local Hausdorff topology to \mathcal{H}^r . A straight-forward computation shows that

$$k_{\mathcal{H}^r}(x; v) = \lim_{n \rightarrow \infty} k_{A_n \Delta^r}(x; v)$$

uniformly on compact sets of $\mathcal{H}^r \times \mathbb{C}^r$ and

$$K_{\mathcal{H}^r}(x, y) = \lim_{n \rightarrow \infty} K_{A_n \Delta^r}(x, y)$$

uniformly on compact sets of $\mathcal{H}^r \times \mathcal{H}^r$.

As a consequence of this discussion we have the following:

Observation 2.5. With the notation above, for any $u \in \mathcal{H}^r$ there exists $u_n \in \Delta^r$ so that

- (1) $\lim_{n \rightarrow \infty} A_n(u_n) = u$,
- (2) $\lim_{n \rightarrow \infty} K_{\Delta^r}(u_n, w_n) = K_{\mathcal{H}^r}(u, (i, \dots, i))$.

In the context of the above observation the fact that

$$\lim_{n \rightarrow \infty} K_{\Delta^r}(u_n, w_n) < \infty$$

implies that

$$\lim_{n \rightarrow \infty} \|u_n - w_n\| = 0.$$

3. LOCAL COORDINATES NEAR A BOUNDARY POINT

In this section we construct useful coordinates around a given boundary point of a generic analytic polyhedron:

Proposition 3.1. *Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron with generic defining functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$. If $\xi \in \partial\Omega$ and*

$$\{i_1, \dots, i_r\} = \{i : |f_i(\xi)| = 1\},$$

then there exists a neighborhood \mathcal{O} of ξ and holomorphic maps $\Psi : \mathcal{O} \rightarrow \mathbb{C}^d$ and $\psi : \mathcal{O} \rightarrow \mathbb{C}^{d-r}$ so that

- (1) Ψ is a biholomorphism onto its image,
- (2) $\Psi(z) = (f_{i_1}(z), \dots, f_{i_r}(z), \psi(z))$ for all $z \in \mathcal{O}$,
- (3) $\Psi(\mathcal{O}) = \mathcal{O}_1 \times \dots \times \mathcal{O}_d$ for some open sets $\mathcal{O}_i \subset \mathbb{C}$,
- (4) $\Psi(\mathcal{O} \cap \Omega) = \Psi(\mathcal{O}) \cap \Delta^d = (\mathcal{O}_1 \cap \Delta) \times \dots \times (\mathcal{O}_r \cap \Delta) \times \mathcal{O}_{r+1} \times \dots \times \mathcal{O}_d$.

Proof. For a neighborhood \mathcal{O} of ξ define the map $F : \mathcal{O} \rightarrow \mathbb{C}^r$ by

$$F(z) = (f_{i_1}(z), \dots, f_{i_r}(z)).$$

By shrinking \mathcal{O} we may assume that

$$\Omega \cap \mathcal{O} = F^{-1}(\Delta^r \cap F(\mathcal{O})).$$

Since the vectors

$$\nabla f_{i_1}(\xi), \dots, \nabla f_{i_r}(\xi)$$

are \mathbb{C} -linearly independent we see that $d(F)_\xi$ has full rank. Hence, after possibly shrinking \mathcal{O} , we can find domains $V \subset \mathbb{C}^r$, $W \subset \mathbb{C}^{d-r}$ and a biholomorphism $\Psi : \mathcal{O} \rightarrow V \times W$ so that

$$F = \pi_1 \circ \Psi$$

where $\pi_1 : V \times W \rightarrow V$ is the natural projection. Thus there exists a holomorphic map $\psi : \mathcal{O} \rightarrow \mathbb{C}^{d-r}$ so that

$$\Psi(z) = (f_{i_1}(z), \dots, f_{i_r}(z), \psi(z))$$

for all $z \in \mathcal{O}$. Moreover

$$\Psi^{-1}((\Delta^r \cap V) \times W) = F^{-1}(\Delta^r \cap F(\mathcal{O})) = \Omega \cap \mathcal{O}.$$

Finally by shrinking \mathcal{O} again we may assume that $\Psi(\mathcal{O}) = \mathcal{O}_1 \times \dots \times \mathcal{O}_d$ for some open sets $\mathcal{O}_i \subset \mathbb{C}$. \square

4. ESTIMATES FOR THE KOBAYASHI METRIC

For the rest of the section suppose that $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron with generic defining functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$.

Using Proposition 2.1 we immediately obtain the following lower bound for the Kobayashi metric:

Proposition 4.1. *With the notation above,*

$$\max_{i=1, \dots, N} k_{\Delta}(f_i(z); d(f_i)_z(v)) \leq k_{\Omega}(z; v).$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$.

In this section we will prove the following upper bound:

Theorem 4.2. *With the notation above, there exists a constant $C \geq 1$ so that*

$$k_{\Omega}(z; v) \leq C \left(\|v\| + \max_{i=1, \dots, N} k_{\Delta}(f_i(z); d(f_i)_z(v)) \right)$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$.

Remark 4.3. With no additional assumptions on the defining functions it is necessary to have the $\|v\|$ term in the estimate above. In particular, it is possible for there to exist some $z_0 \in \Omega$ where

$$\text{Span}_{\mathbb{C}} \{\nabla f_1(z_0), \dots, \nabla f_N(z_0)\} \neq \mathbb{C}^d.$$

Then there would exist some non-zero $v \in \mathbb{C}^d$ so that

$$\max_{i=1, \dots, N} k_{\Delta}(f_i(z_0); d(f_i)_{z_0}(v)) = 0.$$

We begin the proof of Theorem 4.2 by proving a local version:

Lemma 4.4. *With the notation above, for any $\xi \in \partial\Omega$ there exists a neighborhood V of ξ and a constant $C \geq 1$ so that*

$$k_{\Omega}(z; v) \leq C \left(\|v\| + \max_{i \in \mathcal{I}(\xi)} k_{\Delta}(f_i(z); d(f_i)_z(v)) \right)$$

for all $z \in \Omega \cap V$ and $v \in \mathbb{C}^d$.

Proof. After relabeling the f_i , we may assume that $|f_1(\xi)| = \dots = |f_r(\xi)| = 1$ and $|f_i(\xi)| < 1$ when $i > r$. Then

$$\mathcal{I}(\xi) = \{1, \dots, r\}.$$

By Proposition 3.1, there exists a neighborhood \mathcal{O} of ξ and holomorphic maps $\Psi : \mathcal{O} \rightarrow \mathbb{C}^d$ and $\psi : \mathcal{O} \rightarrow \mathbb{C}^{d-r}$ so that:

- (1) Ψ is a biholomorphism onto its image,
- (2) $\Psi(z) = (f_1(z), \dots, f_r(z), \psi(z))$ for all $z \in \mathcal{O}$,
- (3) $\Psi(\mathcal{O}) = \mathcal{O}_1 \times \dots \times \mathcal{O}_d$ for some open sets $\mathcal{O}_i \subset \mathbb{C}$,
- (4) $\Psi(\mathcal{O} \cap \Omega) = \Psi(\mathcal{O}) \cap \Delta^d = (\mathcal{O}_1 \cap \Delta) \times \dots \times (\mathcal{O}_r \cap \Delta) \times \mathcal{O}_{r+1} \times \dots \times \mathcal{O}_d$.

Using Observation 2.2 we can find a neighborhood $V \Subset \mathcal{O}$ of ξ and some $M > 0$ so that

- (1) $\Psi(V) = V_1 \times \dots \times V_d$ for some open sets $V_i \subset \mathbb{C}$,

(2) for $1 \leq i \leq r$ we have

$$k_{\mathcal{O}_i \cap \Delta}(z; v) \leq e^M k_{\Delta}(z; v)$$

for $z \in V_i \cap \Delta$ and $v \in \mathbb{C}$, and

(3) for $r < i$ we have

$$k_{\mathcal{O}_i}(z; v) \leq e^M \|v\|$$

for $z \in V_i$ and $v \in \mathbb{C}$.

Now let Ψ_i be the i^{th} coordinate function of Ψ . Then for $z \in \mathcal{O} \cap \Omega$ and $v \in \mathbb{C}^d$

$$\begin{aligned} k_{\Omega}(z; v) &\leq k_{\mathcal{O} \cap \Omega}(z; v) \\ &= \max \left\{ \max_{i=1, \dots, r} k_{\mathcal{O}_i \cap \Delta}(\Psi_i(z); d(\Psi_i)_z(v)), \max_{i=r+1, \dots, N} k_{\mathcal{O}_i}(\Psi_i(z); d(\Psi_i)_z(v)) \right\}. \end{aligned}$$

For $1 \leq i \leq r$ and $z \in V \cap \Omega$ we have

$$k_{\mathcal{O}_i \cap \Delta}(\Psi_i(z); d(\Psi_i)_z(v)) = k_{\mathcal{O}_i \cap \Delta}(f_i(z); d(f_i)_z(v)) \leq e^M k_{\Delta}(f_i(z); d(f_i)_z(v)).$$

Since $\overline{V} \subset \mathcal{O}$ there exists $C_0 \geq 1$ so that

$$|d(\Psi_i)_z(v)| \leq C_0 \|v\|$$

for all $z \in V$ and $v \in \mathbb{C}^d$. So for $i > r$, $z \in V$, and $v \in \mathbb{C}^d$ we have:

$$k_{\mathcal{O}_i}(\Psi_i(z); d(\Psi_i)_z(v)) \leq e^M \|d(\Psi_i)_z(v)\| \leq C_0 e^M \|v\|.$$

So

$$k_{\Omega}(z; v) \leq C_0 e^M \left(\|v\| + \max_{i=1, \dots, r} k_{\Delta}(f_i(z); d(f_i)_z(v)) \right)$$

for all $z \in V \cap \Omega$ and $v \in \mathbb{C}^d$. □

Proof of Theorem 4.2. Now for every $\xi \in \partial\Omega$ there exists a neighborhood V_{ξ} of ξ and a constant $C_{\xi} \geq 1$ so that

$$k_{\Omega}(z; v) \leq C_{\xi} \left(\|v\| + \max_{i=1, \dots, N} k_{\Delta}(f_i(z); d(f_i)_z(v)) \right)$$

for all $z \in \Omega \cap V_{\xi}$ and $v \in \mathbb{C}^d$. Then since $\partial\Omega$ is compact, there exists $\xi_1, \dots, \xi_k \in \partial\Omega$ so that

$$\partial\Omega \subset \cup_{i=1}^k V_{\xi_i}.$$

Since $K := \Omega \setminus \cup_{i=1}^k V_{\xi_i}$ is compact there exists $C_0 \geq 1$ so that

$$k_{\Omega}(z; v) \leq C_0 \|v\|$$

for all $z \in K$ and $v \in \mathbb{C}^d$.

Finally let $C = \max\{C_{\xi_1}, \dots, C_{\xi_k}, C_0\}$. Then

$$k_{\Omega}(z; v) \leq C \left(\|v\| + \max_{i=1, \dots, N} k_{\Delta}(f_i(z); d(f_i)_z(v)) \right)$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$. □

5. ESTIMATES FOR THE KOBAYASHI DISTANCE

For the rest of the section suppose that $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron with generic defining functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$.

Using Proposition 2.1 we immediately obtain the following lower bound for the Kobayashi distance:

Proposition 5.1. *With the notation above,*

$$\max_{i=1, \dots, N} K_{\Delta}(f_i(z), f_i(w)) \leq K_{\Omega}(z, w)$$

for all $z, w \in \Omega$.

In this section we will establish an upper bound on the Kobayashi distance.

Theorem 5.2. *With the notation above, for any $\xi \in \partial\Omega$ there exists a neighborhood V of ξ and a constant $C \geq 1$ so that*

$$K_{\Omega}(z, w) \leq C \left(\|z - w\| + \max_{i \in \mathcal{I}(\xi)} K_{\Delta}(f_i(z), f_i(w)) \right)$$

for all $z, w \in V \cap \Omega$.

Remark 5.3. In general there will not exist a $C \geq 1$ so that the upper bound

$$K_{\Omega}(z, w) \leq C \left(\|z - w\| + \max_{i=1, \dots, N} K_{\Delta}(f_i(z), f_i(w)) \right)$$

holds for **all** $z, w \in \Omega$. The issue is that for some subset $\{i_1, \dots, i_r\} \subset \{1, \dots, N\}$ and $\zeta_1, \dots, \zeta_r \in \partial\Delta$ the set

$$\{\xi \in \partial\Omega : f_{i_j}(\xi) = \zeta_j \text{ for } 1 \leq j \leq r \text{ and } |f_i(\xi)| < 1 \text{ if } i \notin \{i_1, \dots, i_r\}\}$$

could have multiple components in $\partial\Omega$. But in this case, using Proposition 6.1 below, one could find $z_n, w_n \in \Omega$ so that

$$\limsup_{n \rightarrow \infty} K_{\Omega}(z_n, w_n) = \infty.$$

but

$$\limsup_{n \rightarrow \infty} \left(\max_{i=1, \dots, N} K_{\Delta}(f_i(z_n), f_i(w_n)) \right) < \infty.$$

Proof. After relabeling the f_i , we may assume that $|f_1(\xi)| = \dots = |f_r(\xi)| = 1$ and $|f_i(\xi)| < 1$ when $i > r$. Then

$$\mathcal{I}(\xi) = \{1, \dots, r\}.$$

By Proposition 3.1, there exists a neighborhood \mathcal{O} of ξ and holomorphic maps $\Psi : \mathcal{O} \rightarrow \mathbb{C}^d$ and $\psi : \mathcal{O} \rightarrow \mathbb{C}^{d-r}$ so that:

- (1) Ψ is a biholomorphism onto its image,
- (2) $\Psi(z) = (f_1(z), \dots, f_r(z), \psi(z))$ for all $z \in \mathcal{O}$,
- (3) $\Psi(\mathcal{O}) = \mathcal{O}_1 \times \dots \times \mathcal{O}_d$ for some open sets $\mathcal{O}_i \subset \mathbb{C}$,
- (4) $\Psi(\mathcal{O} \cap \Omega) = \Psi(\mathcal{O}) \cap \Delta^d = (\mathcal{O}_1 \cap \Delta) \times \dots \times (\mathcal{O}_r \cap \Delta) \times \mathcal{O}_{r+1} \times \dots \times \mathcal{O}_d$.

Using Observation 2.2 we can find a neighborhood $V \Subset \mathcal{O}$ of ξ and some $M > 0$ so that

- (1) $\Psi(V) = V_1 \times \dots \times V_d$ for some open sets $V_i \subset \mathbb{C}$,

(2) for $1 \leq i \leq r$ we have

$$K_{\mathcal{O}_i \cap \Delta}(z, w) \leq e^M K_{\Delta}(z, w)$$

for $z, w \in V_i \cap \Delta$, and

(3) for $r < i$ we have

$$K_{\mathcal{O}_i}(z, w) \leq e^M \|z - w\|$$

for $z, w \in V_i$.

Now let Ψ_i be the i^{th} coordinate function of Ψ . Then for $z, w \in \mathcal{O} \cap \Omega$

$$\begin{aligned} K_{\Omega}(z, w) &\leq K_{\mathcal{O} \cap \Omega}(z, w) \\ &= \max \left\{ \max_{i=1, \dots, r} K_{\mathcal{O}_i \cap \Delta}(\Psi_i(z), \Psi_i(w)), \max_{i=r+1, \dots, d} K_{\mathcal{O}_i}(\Psi_i(z), \Psi_i(w)) \right\} \end{aligned}$$

If $1 \leq i \leq r$, then $\Psi_i(z) = f_i(z)$ and so

$$K_{\mathcal{O}_i \cap \Delta}(\Psi_i(z), \Psi_i(w)) \leq e^M K_{\Delta}(f_i(z), f_i(w))$$

for $z, w \in V \cap \Omega$. Since $\overline{V} \subset \mathcal{O}$ there exists $C_0 \geq 1$ so that

$$|\Psi_i(z) - \Psi_i(w)| \leq C_0 \|z - w\|$$

for all $z, w \in V$. So for $i > r$ and $z, w \in V$

$$K_{\mathcal{O}_i}(\Psi_i(z), \Psi_i(w)) \leq C_0 e^M \|z - w\|.$$

So

$$K_{\Omega}(z, w) \leq C_0 e^M \left(\|z - w\| + \max_{i=1, \dots, N} K_{\Delta}(f_i(z), f_i(w)) \right)$$

for all $z, w \in V \cap \Omega$. □

6. THE ASYMPTOTIC GEOMETRY OF GENERIC ANALYTIC POLYHEDRONS

In this section we will prove two facts about the asymptotic geometry of bounded generic analytic polyhedron.

For the rest of the section suppose that $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron with generic defining functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$.

Proposition 6.1. *With the notation above, suppose x_n and y_n are sequences in Ω , $x_n \rightarrow \xi \in \partial\Omega$, $y_n \rightarrow \eta \in \partial\Omega$, and*

$$\limsup_{n \rightarrow \infty} K_{\Omega}(x_n, y_n) < \infty,$$

then $\eta \in \mathcal{F}(\xi)$.

Proof. After relabeling the f_i , we may assume that $|f_1(\xi)| = \dots = |f_r(\xi)| = 1$ and $|f_i(\xi)| < 1$ when $i > r$.

By Proposition 2.4 (Ω, K_{Ω}) is a Cauchy complete length space. Thus every two points in Ω can be joined by a geodesic. Let $\sigma_n : [0, T_n] \rightarrow \Omega$ be a geodesic so that $\sigma_n(0) = x_n$ and $\sigma_n(T_n) = y_n$. Now fix $R > 0$ so that

$$\Omega \Subset B_R := \{z \in \mathbb{C}^d : \|z\| < R\}.$$

Then since $K_{B_R} \leq K_{\Omega}$ on Ω we see that

$$K_{B_R}(\sigma_n(t), \sigma_n(s)) \leq K_{\Omega}(\sigma_n(t), \sigma_n(s)) = |t - s|.$$

So each σ_n is 1-Lipschitz when viewed as a map from $[0, T_n]$ to (B_R, K_{B_R}) . Then since (B_R, K_{B_R}) is a Cauchy complete metric space, see for instance Corollary 2.3.6 in [Aba89], we can pass to a subsequence so that σ_n converges uniformly to a curve $\sigma : [0, T] \rightarrow \overline{\Omega}$ with $\sigma(0) = \xi$ and $\sigma(T) = \eta$. Since

$$K_\Omega(\sigma_n(t), \sigma_n(0)) = t$$

using Proposition 5.1 we see that

$$f_i(\sigma(t)) = \lim_{n \rightarrow \infty} f_i(\sigma_n(t)) = f_i(\xi) \text{ when } 1 \leq i \leq r$$

and

$$|f_i(\sigma(t))| = \lim_{n \rightarrow \infty} |f_i(\sigma_n(t))| < 1 \text{ when } r < i.$$

So the image of σ is contained in

$$\{z \in \partial\Omega : f_i(z) = f_i(\xi) \text{ if } 1 \leq i \leq r \text{ and } |f_i(z)| < 1 \text{ if } i > r\}$$

and hence $\eta \in \mathcal{F}(\xi)$. \square

Proposition 6.2. *With the notation above, suppose $z_0 \in \Omega$, $\varphi_n \in \text{Aut}(\Omega)$, $\varphi_n(z_0) \rightarrow \xi \in \partial\Omega$, and φ_n converges locally uniformly to a holomorphic map $\varphi_\infty : \Omega \rightarrow \overline{\Omega}$. Then $\varphi_\infty(\Omega) = \mathcal{F}(\xi)$.*

Proof. Since

$$\limsup_{n \rightarrow \infty} K_\Omega(\varphi_n(z_0), \varphi_n(z)) = K_\Omega(z_0, z)$$

we see from the previous Proposition that $\varphi_\infty(\Omega) \subset \mathcal{F}(\xi)$.

After relabeling the f_i , we may assume that $|f_1(\xi)| = \dots = |f_r(\xi)| = 1$ and $|f_i(\xi)| < 1$ when $i > r$.

By Proposition 3.1, for each $\eta \in \mathcal{F}(\xi)$ there exists a neighborhood \mathcal{O}_η of η and holomorphic maps $\Psi_\eta : \mathcal{O}_\eta \rightarrow \mathbb{C}^d$ and $\psi_\eta : \mathcal{O}_\eta \rightarrow \mathbb{C}^{d-r}$ so that:

- (1) Ψ_η is a biholomorphism onto its image,
- (2) $\Psi_\eta(z) = (f_1(z), \dots, f_r(z), \psi_\eta(z))$ for all $z \in \mathcal{O}_\eta$,
- (3) $\Psi(\mathcal{O}_\eta) = U_\eta \times W_\eta$ for some open sets $U_\eta \subset \mathbb{C}^r$ and $W_\eta \subset \mathbb{C}^{d-r}$,
- (4) $\Psi(\mathcal{O}_\eta \cap \Omega) = (U_\eta \cap \Delta^r) \times W_\eta$.

Using Theorem 5.2 we may also assume that there exists some $C_\eta \geq 1$ so that

$$K_\Omega(z, w) \leq C_\eta \left(1 + \max_{i=1, \dots, r} K_\Delta(f_i(z), f_i(w)) \right)$$

for $z, w \in \mathcal{O}_\eta \cap \Omega$.

Now suppose that $\eta \in \mathcal{F}(\xi)$. To show that $\eta \in \varphi_\infty(\Omega)$ we need to find a sequence of points $y_n \in \Omega$ so that $y_n \rightarrow \eta$ and

$$\liminf_{n \rightarrow \infty} K_\Omega(\varphi_n^{-1}(y_n), z_0) < \infty.$$

To this end, let $\sigma : [0, 1] \rightarrow \mathcal{F}(\xi)$ be a curve with $\sigma(0) = \xi$ and $\sigma(1) = \eta$. Now we can find $\eta_1, \dots, \eta_m \in \mathcal{F}(\xi)$ so that

$$\sigma([0, 1]) \subset \cup_{j=1}^m \mathcal{O}_{\eta_j}.$$

By relabeling and decreasing the size of our cover, we may assume that $\xi \in \mathcal{O}_{\eta_1}$ and for each $1 \leq j \leq m-1$ there exists some $0 \leq t_j \leq 1$ with

$$\sigma(t_j) \in \mathcal{O}_{\eta_j} \cap \mathcal{O}_{\eta_{j+1}}$$

Next let $u := (f_1, \dots, f_r)(\xi)$ and $u_n := (f_1, \dots, f_r)(\varphi_n(z_0))$. Then for n large we have $u_n \in U_{\eta_j}$ for all $1 \leq j \leq m$.

Let $w_1 \in W_{\eta_1}$ be the unique point so that

$$(u_n, w_1) = \Phi_{\eta_1}(\varphi_n(z_0))$$

and for $2 \leq j \leq m$ let $w_j \in W_{\eta_j}$ be the unique point so that

$$(u, w_j) = \Phi_{\eta_j}(\sigma(t_{j-1})).$$

For $1 \leq j \leq m-1$ let $\bar{w}_j \in W_{\eta_j}$ be the unique point so that

$$(u, \bar{w}_j) = \Phi_{\eta_j}(\sigma(t_j))$$

and let $\bar{w}_m \in W_{\eta_m}$ be the unique point so that

$$(u, \bar{w}_m) = \Phi_{\eta_m}(\eta).$$

Finally let

$$y_n = \Phi_{\eta_m}^{-1}(u_n, \bar{w}_m) \in \Omega.$$

Then by construction

$$\lim_{n \rightarrow \infty} y_n = \eta$$

and

$$\begin{aligned} K_{\Omega}(\varphi_n(z_0), y_n) &\leq \sum_{j=1}^m K_{\Omega} \left(\Phi_{\eta_j}^{-1}(u_n, w_j), \Phi_{\eta_j}^{-1}(u_n, \bar{w}_j) \right) \\ &\leq C_{\eta_1} + \dots + C_{\eta_m}. \end{aligned}$$

So

$$\limsup_{n \rightarrow \infty} K_{\Omega}(\varphi_n^{-1}(y_n), z_0) \leq C_{\eta_1} + \dots + C_{\eta_m}.$$

So by passing to a subsequence we can assume that $\varphi_n^{-1}(y_n)$ converges to some $y \in \Omega$. Then since φ_n converges locally uniformly to φ_{∞} we see that

$$\varphi_{\infty}(y) = \lim_{n \rightarrow \infty} \varphi_n(\varphi_n^{-1}(y_n)) = \eta.$$

Since $\eta \in \mathcal{F}(\xi)$ was arbitrary we see that $\varphi(\Omega) = \mathcal{F}(\xi)$. \square

7. PROOF OF THEOREM 1.9

For the rest of the section suppose that $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron with generic defining functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$ and $\text{Aut}(\Omega)$ is non-compact.

7.1. An embedding. If necessary, we can define holomorphic functions $f_{N+1}, \dots, f_M : U \rightarrow \mathbb{C}^d$ so that

- (1) for any $N+1 \leq i \leq M$, $f_i(\bar{\Omega}) \subset \Delta$,
- (2) for any $z, w \in \bar{\Omega}$ distinct there exists $1 \leq i \leq M$ so that $f_i(z) \neq f_i(w)$,
- (3) for any point $z \in \bar{\Omega}$ we have

$$\text{Span}_{\mathbb{C}} \{ \nabla f_1(z), \dots, \nabla f_M(z) \} = \mathbb{C}^d.$$

Now consider the map $F : \Omega \rightarrow \Delta^M$ given by

$$F(z) = (f_1(z), \dots, f_M(z)).$$

By construction this is a holomorphic embedding of Ω into Δ^M . Moreover, since

$$k_\Delta(f_i(z); d(f_i)_z(v)) = \frac{|d(f_i)_z(v)|}{1 - |f_i(z)|^2} \geq |d(f_i)_z(v)|$$

we see that there exists $\epsilon > 0$ so that

$$k_{\Delta^M}(F(z); d(F)_z(v)) = \max_{i=1, \dots, M} k_\Delta(f_i(z); d(f_i)_z(v)) \geq \epsilon \|v\|$$

for all $z \in \overline{\Omega}$ and $v \in \mathbb{C}^d$.

Then using Theorem 4.2 there exists $C_0 \geq 1$ so that

$$k_{\Delta^M}(F(z); d(F)_z(v)) \leq k_\Omega(z; v) \leq C_0 k_{\Delta^M}(F(z); d(F)_z(v)).$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$.

7.2. Fixing our orbit. Suppose that $\xi \in \mathcal{L}(\Omega)$. Then there exists $w_0 \in \Omega$ and a sequence φ_n in $\text{Aut}(\Omega)$ so that $\varphi_n(w_0) \rightarrow \xi$. Let

$$r = \#\{i : f_i(\xi) = 1\}.$$

By relabeling the functions f_1, \dots, f_N we may assume that

$$\{1, \dots, r\} = \{i : f_i(\xi) = 1\}.$$

Using Proposition 6.2 and possibly passing to a subsequence we can suppose that φ_n converges locally uniformly to a holomorphic map $\varphi_\infty : \Omega \rightarrow \mathcal{F}(\xi)$.

7.3. Constructing affine maps. Let $w_n = \varphi_n(w_0)$. Then for $1 \leq i \leq r$ let

$$\lambda_i^{(n)} = \frac{i}{f_i(w_n)} \frac{|f_i(w_n)|}{|f_i(w_n)| - 1}.$$

Then define affine maps $A_n \in \text{Aff}(\mathbb{C}^r)$ by

$$A_n(z_1, \dots, z_r) = \left(\lambda_1^{(n)} \left(z_1 - \frac{f_1(w_n)}{|f_1(w_n)|} \right), \dots, \lambda_r^{(n)} \left(z_r - \frac{f_r(w_n)}{|f_r(w_n)|} \right) \right).$$

Next define affine maps $\overline{A}_n \in \text{Aff}(\mathbb{C}^M)$ by

$$\overline{A}_n(z_1, \dots, z_M) = (A_n(z_1, \dots, z_r), z_{r+1}, \dots, z_M).$$

Now

$$(\overline{A}_n F \varphi_n)(w_0) = (i, \dots, i, f_{r+1}(w_n), \dots, f_M(w_n))$$

and so $(\overline{A}_n F \varphi_n)(w_0)$ converges to a point $w_\infty \in \mathcal{H}^r \times \Delta^{M-r}$.

7.4. Normal families. By the discussion in Subsection 2.5 we see that $\overline{A}_n(\Delta^M)$ converges in the local Hausdorff topology to $\mathcal{H}^r \times \Delta^{M-r}$. Moreover

$$k_{\mathcal{H}^r \times \Delta^{M-r}}(x; v) = \lim_{n \rightarrow \infty} k_{\overline{A}_n(\Delta^M)}(x; v)$$

uniformly on compact sets of $(\mathcal{H}^r \times \Delta^{M-r}) \times \mathbb{C}^M$ and

$$K_{\mathcal{H}^r \times \Delta^{M-r}}(x, y) = \lim_{n \rightarrow \infty} K_{\overline{A}_n(\Delta^M)}(x, y)$$

uniformly on compact sets of $(\mathcal{H}^r \times \Delta^{M-r}) \times (\mathcal{H}^r \times \Delta^{M-r})$.

Now the map $\Phi_n := (\overline{A}_n F \varphi_n) : \Omega \rightarrow \mathbb{C}^M$ satisfies

$$k_{\overline{A}_n(\Delta^M)}(\Phi_n(z); d(\Phi_n)_z(v)) \leq k_\Omega(z; v) \leq C_0 k_{\overline{A}_n(\Delta^M)}(\Phi_n(z); d(\Phi_n)_z(v))$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$. The lower bound implies that $\Phi_n : \Omega \rightarrow \mathbb{C}^M$ is a normal family. So after passing to a subsequence we can suppose that Φ_n converges locally uniformly to a holomorphic map $\Phi : \Omega \rightarrow \mathcal{H}^r \times \Delta^{M-r}$.

Then

$$k_{\mathcal{H}^r \times \Delta^{M-r}}(\Phi(z); d(\Phi)_z(v)) \leq k_\Omega(z; v) \leq C_0 k_{\mathcal{H}^r \times \Delta^{M-r}}(\Phi(z); d(\Phi)_z(v))$$

for all $z \in \Omega$ and $v \in \mathbb{C}^d$. Notice that the upper bound implies that $\ker d(\Phi)_z = \{0\}$ for every $z \in \Omega$.

Lemma 7.1. $\Phi : \Omega \rightarrow \mathcal{H}^r \times \Delta^{M-r}$ is a biholomorphism onto its image.

Proof. Since $\ker d(\Phi)_z = \{0\}$ for every $z \in \Omega$, the map Φ is a local biholomorphism onto its image. To show that it is a global biholomorphism we need to show that Φ is one-to-one. So suppose that $\Phi(z_1) = \Phi(z_2)$. Then

$$0 = \lim_{n \rightarrow \infty} \overline{A}_n(F(\varphi_n(z_1))) - \overline{A}_n(F(\varphi_n(z_2))).$$

Since $\|\overline{A}_n(z) - \overline{A}_n(w)\| \geq \|z - w\|$ for all n this implies that

$$F(\varphi_\infty(z_1)) = \lim_{n \rightarrow \infty} F(\varphi_n(z_1)) = \lim_{n \rightarrow \infty} F(\varphi_n(z_2)) = F(\varphi_\infty(z_2)).$$

Since F is one-to-one on $\overline{\Omega}$ this implies that $\varphi_\infty(z_1) = \varphi_\infty(z_2)$.

Now by Theorem 5.2 there exists $C_1 \geq 1$ so that for large n we have

$$K_\Omega(\varphi_n(z_1), \varphi_n(z_2)) \leq C_1 \left(\|\varphi_n(z_1) - \varphi_n(z_2)\| + K_{\Delta^M}(F(\varphi_n(z_1)), F(\varphi_n(z_2))) \right).$$

Since $K_\Omega(z_1, z_2) = K_\Omega(\varphi_n(z_1), \varphi_n(z_2))$ for all $n \in \mathbb{N}$ we then see that

$$\begin{aligned} K_\Omega(z_1, z_2) &\leq \lim_{n \rightarrow \infty} C_1 \left(\|\varphi_n(z_1) - \varphi_n(z_2)\| + K_{\Delta^M}(F(\varphi_n(z_1)), F(\varphi_n(z_2))) \right) \\ &= \lim_{n \rightarrow \infty} C_1 K_{\Delta^M}(F(\varphi_n(z_1)), F(\varphi_n(z_2))) \\ &= \lim_{n \rightarrow \infty} C_1 K_{\overline{A}_n(\Delta^M)}(\Phi_n(z_1), \Phi_n(z_2)) = C_1 K_{\mathcal{H}^r \times \Delta^{M-r}}(\Phi(z_1), \Phi(z_2)) = 0. \end{aligned}$$

Thus $z_1 = z_2$ and so Φ is one-to-one. \square

7.5. Analyzing $\Phi(\Omega)$. Let $\pi_2 : \mathbb{C}^r \times \mathbb{C}^{M-r} \rightarrow \mathbb{C}^{M-r}$ be the natural projection and let $W = \pi_2(\Phi(\Omega))$.

Lemma 7.2. $\Phi(\Omega) = \mathcal{H}^r \times W$.

Proof. Since Φ_n converges locally uniformly to Φ , to show that some $y \in \mathbb{C}^M$ is contained in $\Phi(\Omega)$ it is enough to find a sequence $y_n \in \Omega$ so that:

- (1) $\liminf_{n \rightarrow \infty} K_\Omega(y_n, w_0) < \infty$,
- (2) $\lim_{n \rightarrow \infty} \Phi_n(y_n) = y$.

Suppose that $w \in W$. Then there exists $z_0 \in \Omega$ and $u \in \mathcal{H}^r$ so that $\Phi(z_0) = (u, w)$. Consider the sequence $z_n = \varphi_n(z_0)$. Next let

$$\eta := \varphi_\infty(z_0) = \lim_{n \rightarrow \infty} \varphi_n(z_0) = \lim_{n \rightarrow \infty} z_n.$$

Then

$$K_\Omega(w_n, z_n) = K_\Omega(\varphi_n(w_0), \varphi_n(z_0)) = K_\Omega(w_0, z_0)$$

so $\eta \in \mathcal{F}(\xi)$ by Proposition 6.1. In particular,

$$\lim_{n \rightarrow \infty} f_i(w_n) = f_i(\xi) = f_i(\eta) = \lim_{n \rightarrow \infty} f_i(z_n)$$

for $1 \leq i \leq r$ and

$$|f_i(\eta)| = \lim_{n \rightarrow \infty} |f_i(z_n)| < 1$$

for $r < i$.

Now by Proposition 3.1, there exists a neighborhood \mathcal{O} of η and holomorphic maps $\Psi : \mathcal{O} \rightarrow \mathbb{C}^d$ and $\psi : \mathcal{O} \rightarrow \mathbb{C}^{d-r}$ so that:

- (1) Ψ is a biholomorphism onto its image,
- (2) $\Psi(z) = (f_1(z), \dots, f_r(z), \psi(z))$ for all $z \in \mathcal{O}$,
- (3) $\Psi(\mathcal{O}) = V \times W$ for some open sets $V \subset \mathbb{C}^r$ and $W \subset \mathbb{C}^{d-r}$,
- (4) $\Psi(\mathcal{O} \cap \Omega) = \Psi(\mathcal{O}) \cap \Delta^d = (V \cap \Delta^r) \times W$.

Using Theorem 5.2 we may also assume that there exists some $C_2 \geq 1$ so that

$$K_\Omega(z, w) \leq C_2 \left(1 + \max_{i=1, \dots, r} K_\Delta(f_i(z), f_i(w)) \right)$$

for $z, w \in \mathcal{O} \cap \Omega$.

Now suppose that $u' \in \mathcal{H}^r$. Then by Observation 2.5 there exists $u'_n \in \Delta^r$ so that

- (1) $\lim_{n \rightarrow \infty} A_n u'_n = u'$,
- (2) $\limsup_{n \rightarrow \infty} K_{\Delta^r}(u'_n, (f_1, \dots, f_r)(w_n)) < \infty$.

Then

$$\lim_{n \rightarrow \infty} u'_n = \lim_{n \rightarrow \infty} (f_1, \dots, f_r)(w_n) = \lim_{n \rightarrow \infty} (f_1, \dots, f_r)(z_n)$$

So for n large, $u'_n \in V$. Then let

$$z'_n = \Psi^{-1}(u'_n, \psi(z_n)).$$

Then

$$\begin{aligned} K_\Omega(z_n, z'_n) &\leq C_2 (1 + K_{\Delta^r}((f_1, \dots, f_r)(z_n), u'_n)) \\ &\leq C_2 (1 + K_{\Delta^r}((f_1, \dots, f_r)(z_n), (f_1, \dots, f_r)(w_n)) + K_{\Delta^r}((f_1, \dots, f_r)(w_n), u'_n)) \end{aligned}$$

and so

$$\limsup_{n \rightarrow \infty} K_{\Omega}(z_0, \varphi_n^{-1}(z'_n)) = \limsup_{n \rightarrow \infty} K_{\Omega}(z_n, z'_n) < \infty.$$

So after passing to a subsequence we can suppose $\varphi_n^{-1}(z'_n) \rightarrow y \in \Omega$. Then since Φ_n converges locally uniformly to Φ we have

$$\begin{aligned} \Phi(y) &= \lim_{n \rightarrow \infty} \Phi_n(\varphi_n^{-1}(z'_n)) = \lim_{n \rightarrow \infty} (\bar{A}_n F)(z'_n) \\ &= \lim_{n \rightarrow \infty} (\bar{A}_n(u'_n), f_{r+1}(z'_n), \dots, f_M(z'_n)) = (u', w). \end{aligned}$$

Notice that $w = \lim_{n \rightarrow \infty} (f_{r+1}, \dots, f_M)(z'_n)$ since $\lim_{n \rightarrow \infty} \|z_n - z'_n\| = 0$.

Since $u' \in \mathcal{H}^r$ was arbitrary we see that $\mathcal{H}^r \times \{w\} \subset \Phi(\Omega)$. Then since $w \in W$ was arbitrary we see that $\mathcal{H}^r \times W = \Phi(\Omega)$. \square

Lemma 7.3. *W is biholomorphic to $\mathcal{F}(\xi)$.*

Proof. Let $G : \mathcal{F}(\xi) \rightarrow \Delta^{M-r}$ be given by $G(z) = (f_{r+1}, \dots, f_M)(z)$. Since

$$\text{Span}_{\mathbb{C}} \{\nabla f_1(z), \dots, \nabla f_M(z)\} = \mathbb{C}^d$$

for any point $z \in \bar{\Omega}$ and the tangent space of $\mathcal{F}(\xi)$ at z is the orthogonal complement of

$$\text{Span}_{\mathbb{C}} \{\nabla f_1(z), \dots, \nabla f_r(z)\}$$

we see that $\ker d(G)_z = \{0\}$. Thus G is a local biholomorphism. But then using the fact that F is one-to-one on $\bar{\Omega}$ we see that G is one-to-one on $\mathcal{F}(\xi)$. Hence G is a biholomorphism onto its image.

We claim that $G(\mathcal{F}(\xi)) = W$. Let $\pi_2 : \mathbb{C}^r \times \mathbb{C}^{M-r} \rightarrow \mathbb{C}^{M-r}$ be the projection onto the second factor. Then

$$\begin{aligned} \pi_2(\Phi(z)) &= \lim_{n \rightarrow \infty} \pi_2((\bar{A}_n F \varphi_n)(z)) = \lim_{n \rightarrow \infty} (\pi_2 F)(\varphi_n(z)) \\ &= \lim_{n \rightarrow \infty} G(\varphi_n(z)) = G(\varphi_{\infty}(z)). \end{aligned}$$

So $W = \pi_2(\Phi(\Omega)) = G(\varphi_{\infty}(\Omega)) = G(\mathcal{F}(\xi))$ by Proposition 6.2. \square

8. PROOF OF PROPOSITION 1.10

In this section we prove Proposition 1.10. The key step is proving the following refinement of Theorem 1.9:

Theorem 8.1. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded generic analytic polyhedron. If $\xi \in \mathcal{L}(\Omega)$, then there exists a biholomorphism $\Phi : \Omega \rightarrow \Delta^{r(\xi)} \times \mathcal{F}(\xi)$ so that: if $\theta \in \text{Aut}(\mathcal{F}(\xi))$ and $\hat{\theta} = \Phi^{-1}(\text{id}, \theta)\Phi$ then*

$$f_i(\hat{\theta}(z)) = f_i(z)$$

for all $i \in \mathcal{I}(\xi)$ and $z \in \Omega$.

Delaying the proof of Theorem 8.1 let us use it to prove Proposition 1.10:

Proof of Proposition 1.10. Suppose that $\xi_0 \in \mathcal{L}(\Omega)$ and

$$r(\xi_0) = \max\{r(\xi) : \xi \in \mathcal{L}(\Omega)\}.$$

Let $r_0 := r(\xi_0)$. By relabeling our defining functions we can assume that

$$|f_1(\xi_0)| = \cdots = |f_{r_0}(\xi_0)| = 1$$

and $|f_i(\xi_0)| < 1$ when $i > r_0$.

By Theorem 8.1 there exists a biholomorphism $\Phi : \Omega \rightarrow \Delta^{r_0} \times \mathcal{F}(\xi_0)$ so that: if $\theta \in \text{Aut}(\mathcal{F}(\xi_0))$ and $\widehat{\theta} = \Phi^{-1}(\text{id}, \theta)\Phi$ then

$$f_i(\widehat{\theta}(z)) = f_i(z)$$

for all $1 \leq i \leq r_0$ and $z \in \Omega$.

Now suppose for a contradiction that $\text{Aut}(\mathcal{F}(\xi_0))$ is non-compact. Then there exists $\theta_n \in \text{Aut}(\mathcal{F}(\xi_0))$ so that $\theta_n \rightarrow \infty$. Let $\widehat{\theta}_n = \Phi^{-1}(\text{id}, \theta_n)\Phi$, then $\widehat{\theta}_n \rightarrow \infty$ in $\text{Aut}(\Omega)$.

Now fix some sequence $\varphi_n \in \text{Aut}(\Omega)$ and some $z_0 \in \Omega$ so that $\varphi_n(z_0) \rightarrow \xi_0$. Since $\widehat{\theta}_n \rightarrow \infty$ in $\text{Aut}(\Omega)$ there exists some $1 \leq i_0 \leq N$ so that

$$\limsup_{n \rightarrow \infty} |f_{i_0}(\widehat{\theta}_n(z_0))| = 1.$$

Since $f_i(\widehat{\theta}_n(z_0)) = f_i(z_0)$ for $1 \leq i \leq r_0$ we see that $i_0 > r_0$.

Now for each $k \in \mathbb{N}$ pick n_k so that

$$\min_{i=1, \dots, r_0} |f_i(\varphi_{n_k}(z_0))| > 1 - 1/k$$

and then pick m_k so that

$$|f_{i_0}(\widehat{\theta}_{m_k} \varphi_{n_k}(z_0))| > 1 - 1/k.$$

This second choice is possible because of Proposition 6.1.

Notice that

$$\min_{i=1, \dots, r_0} |f_i(\widehat{\theta}_{m_k} \varphi_{n_k}(z_0))| = \min_{i=1, \dots, r_0} |f_i(\varphi_{n_k}(z_0))| > 1 - 1/k.$$

Then by passing to a subsequence we can suppose that $\widehat{\theta}_{m_k} \varphi_{n_k}(z_0) \rightarrow \eta \in \partial\Omega$. Then $\eta \in \mathcal{L}(\Omega)$ and

$$\mathcal{I}(\eta) \supset \{1, \dots, r_0, i_0\}.$$

So

$$r(\eta) > r_0 = \max\{r(\xi) : \xi \in \mathcal{L}(\Omega)\}.$$

and we have a contradiction. \square

8.1. Characteristic decompositions. Before starting the proof of Theorem 8.1 we will need to recall some classical facts about the characteristic decompositions of an analytic polyhedron.

Suppose that $\Omega \subset \mathbb{C}^d$ is a domain and $f : \Omega \rightarrow \mathbb{C}$ is holomorphic. Then for $z \in \Omega$ let $L(z, f)$ be the connected component of $f^{-1}(f(z))$ which contains z . In the case in which Ω is an analytic polyhedron with defining functions $f_1, \dots, f_N : U \rightarrow \Delta$ the decomposition of Ω into sets of the form $L(z, f_i)$ is called a *characteristic decomposition* of Ω .

The following theorem is classical:

Theorem 8.2. [RS60, Satz 14] *Suppose that Ω is an analytic polyhedron and $f_1, \dots, f_N : U \rightarrow \Omega$ is a minimal set of defining functions (that is, any proper subset of f_1, \dots, f_N is not a set of defining functions for Ω). If $\varphi \in \text{Aut}(\Omega)$, then there exists a map $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ so that*

$$\varphi(L(z, f_i)) = L(\varphi(z), f_{\sigma(i)})$$

for all $z \in \Omega$ and $1 \leq i \leq N$.

Because the argument is short we will provide the proof of Theorem 8.2 in Appendix A.

It is possible for different f_i to generate the same characteristic decomposition and so in general the map σ is not uniquely defined. This lack of uniqueness can be overcome by considering a special subset. In particular, fix a subset

$$\{i_1, \dots, i_{n_0}\} \subset \{1, \dots, N\}$$

so that:

- (1) for any $1 \leq i \leq N$ there exists a $1 \leq k \leq n_0$ so that

$$L(z, f_i) = L(z, f_{i_k})$$

for all $z \in \Omega$,

- (2) for all $1 \leq k_1 < k_2 \leq n_0$ there exists some $z \in \Omega$ so that

$$L(z, f_{i_{k_1}}) \neq L(z, f_{i_{k_2}}).$$

Then for any element $\varphi \in \text{Aut}(\Omega)$ there exists a unique map $\sigma(\varphi) : \{1, \dots, n_0\} \rightarrow \{1, \dots, n_0\}$ so that

$$\varphi(L(z, f_{i_k})) = L(\varphi(z), f_{\sigma(\varphi)(i_k)})$$

for any $z \in \Omega$ and $1 \leq k \leq n_0$. Using the uniqueness of $\sigma(\varphi)$, it is straight-forward to verify that σ is a homomorphism from $\text{Aut}(\Omega)$ to the symmetric group $\text{Sym}(n_0)$ on n_0 elements. This leads to the following corollary:

Corollary 8.3. *Suppose that Ω is an analytic polyhedron and $f_1, \dots, f_N : U \rightarrow \Omega$ is a minimal set of defining functions. Then there exists a finite index normal subgroup $H \leq \text{Aut}(\Omega)$ so that*

$$\varphi(L(z, f_i)) = L(\varphi(z), f_i)$$

for all $\varphi \in H$, $z \in \Omega$, and $1 \leq i \leq N$.

8.2. Proof of Theorem 8.1. Suppose Ω is a generic analytic polyhedron and $f_1, \dots, f_N : U \rightarrow \Omega$ is a set of generic defining functions. We may assume that for every $1 \leq i \leq N$ the set

$$\{\xi \in \partial\Omega : |f_i(\xi)| = 1\}$$

is non-empty. Then Proposition 3.1 implies that f_1, \dots, f_N is a minimal set of defining functions. Then, by Corollary 8.3, there exists a finite index normal subgroup $H \leq \text{Aut}(\Omega)$ so that

$$\varphi(L(z, f_i)) = L(\varphi(z), f_i)$$

for all $\varphi \in H$, $z \in \Omega$, and $1 \leq i \leq N$.

Now fix some $\xi \in \mathcal{L}(\Omega)$. Let $r = r(\xi)$ and by relabeling the functions f_1, \dots, f_N we may assume that

$$\{1, \dots, r\} = \{i : |f_i(\xi)| = 1\}.$$

Lemma 8.4. *There exists $w_0 \in \Omega$ and a sequence $\varphi_n \in H$ so that*

$$\varphi_n(w_0) \rightarrow \xi.$$

Proof. Fix some $w'_0 \in \Omega$ and a sequence $\phi_n \in \text{Aut}(\Omega)$ so that

$$\phi_n(w'_0) \rightarrow \xi.$$

By possibly passing to a subsequence we can suppose that ϕ_n converges locally uniformly to a holomorphic map $\phi_\infty : \Omega \rightarrow \overline{\Omega}$.

Consider the natural homomorphism $\rho : \text{Aut}(\Omega) \rightarrow \text{Aut}(\Omega)/H$ and let n_0 be the order of $\text{Aut}(\Omega)/H$. By passing to a subsequence we can suppose that $\rho(\phi_n)$ is constant. Then let

$$\theta = \prod_{i=1}^{n_0-1} \varphi_i$$

and $\varphi_n = \phi_n \theta$. Then φ_n converges locally uniformly to $\phi_\infty \theta$ and so $\varphi_n(w_0) \rightarrow \xi$ where $w_0 = \theta^{-1}(w'_0)$. Moreover,

$$\rho(\varphi_n) = \rho(\phi_n) \prod_{i=1}^{n_0-1} \rho(\phi_i) = \rho(\phi_n)^{n_0} = \text{id}$$

and so $\varphi_n \in H$. □

Now we repeat the proof of Theorem 1.9 with the point $w_0 \in \Omega$ and the sequence $\varphi_n \in H$. In particular: if necessary, we can define holomorphic functions $f_{N+1}, \dots, f_M : U \rightarrow \mathbb{C}^d$ so that

- (1) for any $N+1 \leq i \leq M$, $f_i(\overline{\Omega}) \subset \Delta$,
- (2) for any $z, w \in \overline{\Omega}$ distinct there exists $1 \leq i \leq M$ so that $f_i(z) \neq f_i(w)$,
- (3) for any point $z \in \overline{\Omega}$ we have

$$\text{Span}_{\mathbb{C}} \{\nabla f_1(z), \dots, \nabla f_M(z)\} = \mathbb{C}^d.$$

Now consider the map $F : \Omega \rightarrow \Delta^M$ given by

$$F(z) = (f_1(z), \dots, f_M(z)).$$

Then let $w_n = \varphi_n(w_0)$. Then for $1 \leq i \leq r$ let

$$\lambda_i^{(n)} = \frac{i}{f_i(w_n)} \frac{|f_i(w_n)|}{|f_i(w_n)| - 1}.$$

Then define affine maps $A_n \in \text{Aff}(\mathbb{C}^r)$ by

$$A_n(z_1, \dots, z_r) = \left(\lambda_1^{(n)} \left(z_1 - \frac{f_1(w_n)}{|f_1(w_n)|} \right), \dots, \lambda_r^{(n)} \left(z_r - \frac{f_r(w_n)}{|f_r(w_n)|} \right) \right).$$

Finally define affine maps $\overline{A}_n \in \text{Aff}(\mathbb{C}^M)$ by

$$\overline{A}_n(z_1, \dots, z_M) = (A_n(z_1, \dots, z_r), z_{r+1}, \dots, z_M).$$

As in the proof of Theorem 1.9, we can pass to a subsequence so that the maps $\Phi_n = \overline{A}_n F \varphi_n : \Omega \rightarrow \mathcal{H}^r \times \Delta^{M-r}$ converge to a biholomorphism $\Phi : \Omega \rightarrow \mathcal{H}^r \times W$ where W is biholomorphic to $\mathcal{F}(\xi)$.

Now for $1 \leq i \leq r$ let $\Phi_n^{(i)}$ be the i^{th} coordinate function of Φ_n and let $\Phi^{(i)}$ be the i^{th} coordinate function of Φ . Consider the affine maps $\ell_n^{(i)} \in \text{Aff}(\mathbb{C})$ given by

$$\ell_n^{(i)}(z) = \lambda_i^{(n)} \left(z - \frac{f_i(w_n)}{|f_i(w_n)|} \right).$$

Then $\Phi_n^{(i)} = \ell_n^{(i)} \circ f_i \circ \varphi_n$.

Now if $w \in L(z, f_i)$ and $1 \leq i \leq r$ then

$$\varphi_n(w) \in \varphi_n(L(z, f_i)) = L(\varphi_n(z), f_i)$$

and so $f_i(\varphi_n(w)) = f_i(\varphi_n(z))$. Then we have

$$\Phi^{(i)}(w) = \lim_{n \rightarrow \infty} (\ell_n^{(i)} \circ f_i \circ \varphi_n)(w) = \lim_{n \rightarrow \infty} (\ell_n^{(i)} \circ f_i \circ \varphi_n)(z) = \Phi^{(i)}(z).$$

Thus

$$L(z, f_i) \subset L(z, \Phi^{(i)})$$

for all $z \in \Omega$ and $1 \leq i \leq r$. This implies that

$$\nabla f_i(z) \wedge \nabla \Phi^{(i)}(z) = 0$$

for all $z \in \overline{\Omega}$. So by Lemma A.2 we have:

$$L(z, f_i) = L(z, \Phi^{(i)})$$

for all $z \in \Omega$ and $1 \leq i \leq r$.

Finally suppose that $\theta \in \text{Aut}(\mathcal{F}(\xi))$ and $\widehat{\theta} = \Phi^{-1}(\text{id}, \theta)\Phi$. Then

$$\widehat{\theta} \left(L(z, \Phi^{(i)}) \right) = L(z, \Phi^{(i)})$$

for $1 \leq i \leq r$ and $z \in \Omega$. Hence

$$\widehat{\theta}(L(z, f_i)) = L(z, f_i)$$

and

$$f_i(\widehat{\theta}(z)) = f_i(z)$$

for all $1 \leq i \leq r$ and $z \in \Omega$.

APPENDIX A. CHARACTERISTIC DECOMPOSITIONS

Given a domain $\Omega \subset \mathbb{C}^d$ and a function $f : \Omega \rightarrow \mathbb{C}^d$ let $L(z, f)$ be the connected component of $f^{-1}(f(z))$ which contains z . In this appendix we provide a proof of the following classical result:

Theorem A.1. [RS60, Satz 14] *Suppose Ω is an analytic polyhedron and $f_1, \dots, f_N : U \rightarrow \Omega$ is a minimal defining set. Then for any $\varphi \in \text{Aut}(\Omega)$ there exists a map $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ so that*

$$\varphi(L(z, f_i)) = L(\varphi(z), f_{\sigma(i)})$$

for all $z \in \Omega$ and $1 \leq i \leq N$.

We begin by proving the following lemma:

Lemma A.2. *Suppose $\Omega \subset \mathbb{C}^d$ is a domain and $f, g : \Omega \rightarrow \mathbb{C}$ are non-constant holomorphic functions. If*

$$\nabla f(z) \wedge \nabla g(z) = 0$$

for all $z \in \Omega$, then

$$L(z, f) = L(z, g)$$

for all $z \in \Omega$.

Proof. Fix some $z_0 \in \Omega$ and let A be an irreducible component of $L(z_0, f)$. Then let

$$A_0 = A \cap L(z_0, f)_{reg}$$

where $L(z_0, f)_{reg}$ is the regular points of $L(z_0, f)$. By the Theorem in [Chi89, Chapter 1.5.4], A_0 is connected and open, dense in A .

We claim that $A \subset L(z, g)$ for any $z \in A$. Since A is an irreducible component of $L(z_0, f)$ either $d(g)_z \equiv 0$ on A or $\{z \in A : d(g)_z \neq 0\}$ is an open, dense subset of A . In the first case, clearly g is constant on A and hence $A \subset L(z, g)$ for any $z \in A$. So assume that $\{z \in A : d(g)_z \neq 0\}$ is an open, dense subset of A and fix some $w_0 \in A_0$ so that $d(g)_{w_0} \neq 0$. Then, since $\nabla f(z)$ and $\nabla g(z)$ are co-linear, there exists a neighborhood W of w_0 and a function $h : W \rightarrow \mathbb{C}$ so that

$$d(f)_w = h(w)d(g)_w \text{ for all } w \in W.$$

This implies that $L(w, g|_W) \subset L(w, f|_W)$ for all $w \in W$. Now by assumption w_0 is a regular point of $L(w_0, f) = L(z_0, f)$ and so after possibly shrinking W we may assume that $L(w_0, f|_W)$ is a submanifold of W . Since $L(w_0, g|_W) \subset L(w_0, f|_W)$ Proposition 2 in [Chi89, Chapter 1.2.3] implies that w_0 is a regular point of $L(w_0, g|_W)$. Hence, by possibly shrinking W again, we must have that $L(w_0, g|_W) = L(w_0, f|_W)$. But this implies that g is constant on $A_0 \cap W = L(w_0, f|_W)$. So, since A is irreducible, we see that g is constant on A . Thus $A \subset L(w_0, g)$. Since A is connected, $A \subset L(z, g)$ for any $z \in A$.

Then by the local Noetherian property for analytic sets, see the Theorem in [Chi89, Chapter 1.5.4], there exists a countable set $\{z_\alpha\}$ so that

$$L(z_0, f) \subset \cup L(z_\alpha, g).$$

Then g takes on at most countable many values on $L(z_0, f)$ so connectivity implies that g is constant on $L(z_0, f)$ and hence $L(z_0, f) \subset L(z_0, g)$.

Since $z_0 \in \Omega$ was arbitrary we see that

$$L(z, f) \subset L(z, g)$$

for all $z \in \Omega$. Then by reversing the role of f and g in the argument above we see that

$$L(z, f) = L(z, g)$$

for all $z \in \Omega$. □

The following argument is the proof of Theorem 1.3 in [Zai98] taken essentially verbatim:

Proof of Theorem A.1. Fix some $1 \leq i_0 \leq N$. We will find an $1 \leq i_0^* \leq N$ so that

$$\varphi(L(z, f_{i_0})) = L(\varphi(z), f_{i_0^*})$$

for all $z \in \Omega$.

Since the defining set is minimal there exists some $\xi' \in \partial\Omega$ so that

$$\{i_0\} = \{i : |f_i(\xi')| = 1\}.$$

Then there exists a neighborhood \mathcal{O} of ξ' so that for all $\eta \in \mathcal{O}$ we have

$$\{i_0\} = \{i : |f_i(\eta)| = 1\}.$$

Since f_{i_0} is holomorphic the set

$$\{u \in \mathcal{O} : \nabla f_{i_0}(u) \neq 0\}$$

is connected, see Proposition 3 in [Chi89, Chapter 1.2.2]. Thus there exists some $\xi \in \partial\Omega$ so that

$$\{i_0\} = \{i : |f_i(\xi)| = 1\}$$

and $\nabla f_{i_0}(\xi) \neq 0$.

Then using the proof of Proposition 3.1 there exists a domain $W_1 \subset \mathbb{C}$ containing $f_{i_0}(\xi)$, a domain $W_2 \subset \mathbb{C}^{d-1}$, a neighborhood \mathcal{O} of ξ , and a biholomorphism $\Psi : W_1 \times W_2 \rightarrow \mathcal{O}$ so that $\Psi^{-1}(\Omega) = (\Delta \cap W_1) \times W_2$ and the 1st coordinate function of Ψ^{-1} is f_{i_0} .

Next consider the holomorphic map $G : (\Delta \cap W_1) \times W_2 \rightarrow \otimes_{j=1}^N \mathbb{C}^{d-1}$ given by

$$G(z_1, z_2) = \otimes_{j=1}^N \frac{\partial(f_j \circ \varphi \circ \Psi)}{\partial z_2}(z_1, z_2).$$

We claim that

$$\lim_{z \rightarrow \eta} G(z) = 0$$

for any $\eta \in (\partial\Delta \cap W_1) \times W_2$. Suppose not, then there exists

$$\eta = (\eta_1, \eta_2) \in (\partial\Delta \cap W_1) \times W_2$$

and $z^{(n)} = (z_1^{(n)}, z_2^{(n)}) \rightarrow \eta$ so that

$$\lim_{n \rightarrow \infty} G(z^{(n)}) \neq 0.$$

Consider the sequence of holomorphic maps $\overline{\varphi}_n : W_2 \rightarrow \Omega$ given by

$$\overline{\varphi}_n(z) = \varphi\left(\Psi\left(z_1^{(n)}, z\right)\right).$$

Since Ω is bounded we can pass to a subsequence and assume that $\overline{\varphi}_n$ converges locally uniformly to a holomorphic map $\overline{\varphi} : W_2 \rightarrow \overline{\Omega}$. Since φ is an automorphism and hence proper, we see that $\overline{\varphi}(W_2) \subset \partial\Omega$. Then since

$$\partial\Omega \subset \cup_{j=1}^N f_j^{-1}(\partial\Delta)$$

there exists an open set $W_2' \subset W_2$ and some $1 \leq j_0 \leq N$ so that $\overline{\varphi}(W_2') \subset f_{j_0}^{-1}(\partial\Delta)$. Now $f_{j_0} \circ \overline{\varphi} : W_2 \rightarrow \mathbb{C}$ is holomorphic and $(f_{j_0} \circ \overline{\varphi})(W_2') \subset \partial\Delta$, so $f_{j_0} \circ \overline{\varphi} : W_2 \rightarrow \mathbb{C}$ must be constant. Thus

$$\frac{\partial(f_{j_0} \circ \overline{\varphi})}{\partial z}(z) = 0$$

but then

$$0 = \frac{\partial(f_{j_0} \circ \overline{\varphi})}{\partial z}(z_2) = \lim_{n \rightarrow \infty} \frac{\partial(f_{j_0} \circ \varphi \circ \Psi)}{\partial z_2}(z_1^{(n)}, z_2^{(n)})$$

which is a contradiction. Thus

$$\lim_{z \rightarrow \eta} G(z) = 0$$

for any $\eta \in (\partial\Delta \cap W_1) \times W_2$.

Next we extend G to a map on all of $W_1 \times W_2$ by defining $G(z_1, z_2) = 0$ when $z_1 \in (W_1 \setminus \Delta) \times W_2$. Then the above claim shows that G is continuous and by definition G is holomorphic on $\{z : G(z) \neq 0\}$. So by Rado's Theorem, see [Nar71, pg. 51], G is holomorphic. But G vanishes on an open set, namely $(W_1 \setminus \Delta) \times W_2$, and hence is identically zero. So there exists $1 \leq i_0^* \leq N$ so that

$$\frac{\partial(f_{i_0^*} \circ \varphi \circ \Psi)}{\partial z_2}(z_1, z_2) = 0$$

on $(\Delta \cap W_1) \times W_2$.

Then $\nabla f_{i_0} \wedge \nabla(f_{i_0^*} \circ \varphi) = 0$ on U and so by analyticity $\nabla f_{i_0} \wedge \nabla(f_{i_0^*} \circ \varphi) = 0$ on Ω . So by the Lemma

$$L(z, f_{i_0}) = L(z, f_{i_0^*} \circ \varphi)$$

for all $z \in \Omega$. Thus

$$\varphi(L(z, f_{i_0})) = L(\varphi(z), f_{i_0} \circ \varphi^{-1}) = L(\varphi(z), f_{i_0^*})$$

for all $z \in \Omega$. □

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